Journal of Directed Energy, **2**, Fall 2007, 312–324

Kalman Estimation of Anisoplanatic Zernike Tilt

Todd M. Venema* and Juan R. Vasquez[1]
Air Force Institute of Technology, 2950 Hobson Way,
Wright–Patterson Air Force Base, Ohio 45433

Anisoplanatism causes wavefront estimation errors when compensating for atmospheric turbulence of distant, fast-moving objects using wavefronts received from the object to measure the turbulence. An excellent example of this is the case of using adaptive optics for imaging or communication with satellites in low Earth orbit. By the time the light has made a round-trip from the satellite to the ground and back, the satellite will have moved approximately 50 μrad. Linear estimation (extrapolation) of wavefront tilt parameters has been shown to mitigate anisoplanatism, providing significant improvement in a noise-free environment. We present Kalman filter estimation in lieu of simple linear estimation and demonstrate the robustness of this new approach.

KEYWORDS: Anisoplanatic estimation, Kalman filter, Laser communication

1. Introduction

The goal of this paper is to demonstrate the effectiveness of using a Kalman filter to predict atmospheric tilts in a look-ahead situation. The research builds upon a technical note from David Fried, in which he uses a linear estimator to predict the look-ahead tilt needed for a ground-to-low-Earth-orbit (LEO) satellite system. Fried measures the tilt of a wavefront received from a target LEO satellite and computes an estimate of the tilt compensation needed to minimize atmospheric interference when communicating back to the satellite.[2] His estimate is based on the current tilt received plus the rate of change of that tilt multiplied by the amount of time it takes light to make a round-trip between the satellite and the ground. Rather than using simple linear estimation (i.e., extrapolation), this paper uses a Kalman filter to estimate the required tilt. Performance is measured by estimation error variance. The cases of no estimation, linear estimation, and Kalman estimation are compared.

A WaveTrain® simulation is developed that models a wavefront received from a point source on a LEO satellite. A noiseless sensor detects and records the light's complex field in the aperture plane. The simulations emulate conditions used by Fried and are used to develop time histories of Zernike tilt (ztilt) similar to his. Next, ztilt caused by atmospheric turbulence is modeled as a stochastic process so that a Kalman filter can track and estimate future ztilts. A key element of the modeling is matching the temporal bandwidth of the stochastic process with the temporal bandwidth of the effects from turbulence. Finally the

Received December 7, 2006; revision received August 21, 2007.
*Corresponding author; e-mail: Todd.Venema@AFIT.edu.
[1]E-mail: Juan.Vasquez@AFIT.edu.

simulation and Kalman filter are used to compare the various estimation methods under ideal conditions.

After demonstrating the Kalman estimator's effectiveness under ideal conditions, its effectiveness under nonideal conditions is demonstrated by simulating a measured tilt derived from a Shack–Hartmann (S–H) wavefront sensor (WFS). These more realistic tilt measurements are then input to the Kalman filter and used to estimate ztilt look-ahead.

2. Background

2.1. Adaptive optics

Adaptive optics (AO) is a process of compensating for atmospheric turbulence. First the effects of atmospheric turbulence are measured. For simulation purposes, the satellite is assumed to be cooperative and to have an idealized point source beacon on it. The beacon's light will propagate down to the ground receiver, where a field sensor will detect the light's complex field. In practice there is no such thing as a field sensor, so a WFS would be used to provide the wavefront tilt at each subaperture. Once the atmospheric effect has been measured, it can be corrected via the AO. Higher order modes are compensated for with a deformable mirror. Tilt in the x and y directions (Zernike modes 2 and 3, sometimes called tip and tilt) are generated using a fast-steering mirror. The effects of turbulence are mitigated, giving greatly improved performance.

2.2. Tilt

There are several types of tilt. The simplest is centroid tilt (ctilt), which is equivalent to focusing incoming light and tracking the centroid of the focused beam by[3]

$$\text{centroid}_x = \frac{1}{\sum I_i} \sum x_i I_i, \tag{1}$$

where x_i is the position of the ith pixel in the aperture and I_i is the intensity of the light on the ith pixel. This centroid measurement can be converted into ctilt by knowing the focal length of the focusing lens using

$$\text{ctilt}_x = \tan^{-1}\left(\frac{\text{centroid}_x}{\text{focal length}}\right). \tag{2}$$

This is the type of tilt one would measure and correct for if one did not have a deformable mirror to correct higher order modes and had only a steering mirror to correct the tip and tilt of the system. Gradient tilt (gtilt) can be calculated using a WFS. A typical WFS is a Shack–Hartmann sensor, which divides the aperture into a large number of subapertures. The light on these subapertures is focused onto a detector, which can track the centroid of each subaperture's focused beam. The ctilt of each subaperture can then be averaged to determine the gtilt of the overall aperture[3] using

$$\text{gtilt}_x = \frac{1}{\sum i} \sum \text{ctilt}_{x_i}, \tag{3}$$

where ctilt_{x_i} is the ctilt in the x direction of the ith subaperture. ctilt is equivalent to gtilt for the extreme case in which the aperture has a single subaperture. ctilt can also be determined by weighting the tilts of smaller subapertures by the intensities on those subapertures. ztilt is the average tilt over the entire aperture. This is computed as[2]

$$\text{ztilt}_x = \frac{\alpha}{\sum x_i^2} \sum \phi_i x_i, \tag{4}$$

where $\alpha = (\lambda/2\pi)\Delta x$, λ is the wavelength, Δx is the pixel spacing, ϕ_i is the phase of the ith pixel in the aperture, and x_i is the position of the ith pixel.

The relationship between ctilt, gilt, and ztilt is dependant on the aperture size D of the optical system to the atmospheric coherence diameter r_0. For small D/r_0, ctilt approaches ztilt and is equivalent to ztilt under no-turbulence conditions in which r_0 is infinite and D/r_0 is zero.[4] In the case of a S–H WFS, the diameter D_{SA} of a single lenslet (or subaperture) in the S–H lenslet array is the relevant aperture diameter. Once the subapertures are sized so that D_{SA}/r_0 is small enough that the ctilt of a subaperture is effectively the ztilt of that subaperture, the gtilt calculated from the array of subapertures becomes the ztilt of the entire aperture. Interestingly, the ctilt of the entire aperture can be computed from the ctilts of the S–H subapertures by

$$\text{ctilt}_x = \frac{1}{\sum I_i} \sum I_i \text{ctilt}_{x_i}. \tag{5}$$

This is essentially the same as Eq. (3) for gtilt except that the subapertures are weighted by their intensities.

2.3. Anisoplanatism

Conceptually, the optical path between a transmitter and receiver is a tube of air. An AO system compensates for the turbulence in that tube of air. The tube of air where the turbulence effects are measured is never exactly the same as the tube of air through which the system transmits, but if the two tubes are close enough that the atmospheres in them are the same (unchanged), the system is considered isoplanatic. Otherwise the system is anisoplanatic. The isoplanatic angle θ_0 is the angle over which the atmosphere is essentially unchanged and is given by[1]

$$\theta_0 = \frac{\cos^{8/5}(\zeta)}{\left[2.91k^2 \int_{h_0}^{H} C_n^2(h)(h - h_0)^{5/3}\text{d}h\right]^{3/5}}, \tag{6}$$

where ζ is zenith angle, k is $\lambda/2\pi$, h_0 and H are the ground and target heights, and $C_n^2(h)$ is the refractive-index structure constant as a function of height. The isoplanatic angle under the specified conditions is 12.7 μrad.

The angular difference between the optical paths where the turbulence is measured and the system transmits can be determined by the geometry of the system. The satellite is orbiting at a height of 500 km. Light from the satellite's point beacon travels down to the ground receiver. The atmospheric correction is instantaneously measured and put onto the deformable mirror, and a signal is sent from the ground transmitter to the satellite. This takes $\Delta T = (2)\text{path length}/c = 3.3356$ ms. During this time, the satellite has moved $\Delta x = v\Delta T = 23.86$ m. The angular path difference is 23.86 m/500 km = 47.73 μrad. As this angular difference is significantly greater than the isoplanatic angle, the system is clearly anisoplanatic.

2.4. Kalman filter

Kalman filters are mathematical tools used to estimate a system.[5] Essentially they accept inputs, apply those inputs to a model, and output estimates of system behavior. They are widely known to be "optimal" estimators, and within the constraints of how they are designed, they are. Their biggest limitation is in the accuracy of the models that they use or, in other words, how much the system works like the filter thinks it does.[5] Our model of turbulence effects is that ztilt is a zero mean Gaussian distribution. Thus the ztilt in both the z and y directions will have a Gaussian probability distribution function around zero. The variance of the distribution is proportional to the strength of the turbulence. In our case, we are not so much concerned with the variance of the ztilt, but with how quickly it changes. We are trying to predict what the ztilt will be in 3.3356 ms. The temporal variance of the ztilt is driven by how the turbulence changes. On a millisecond level, the atmosphere is nearly constant (or frozen). Thus if the optical path was stationary (perhaps we were using laser communication with a geostationary satellite), the temporal ztilt variance would be very low. Our optical path, however, moves quickly through the atmospheric turbulence as we are working with a fast-moving LEO satellite. Thus, we expect the speed at which the tilt changes to be proportional to both the strength of the turbulence (which is effectively a scaling factor) and the speed at which the path moves through the atmosphere (which is fixed) and inversely proportional to the aperture diameter, which controls the width of the atmospheric path.

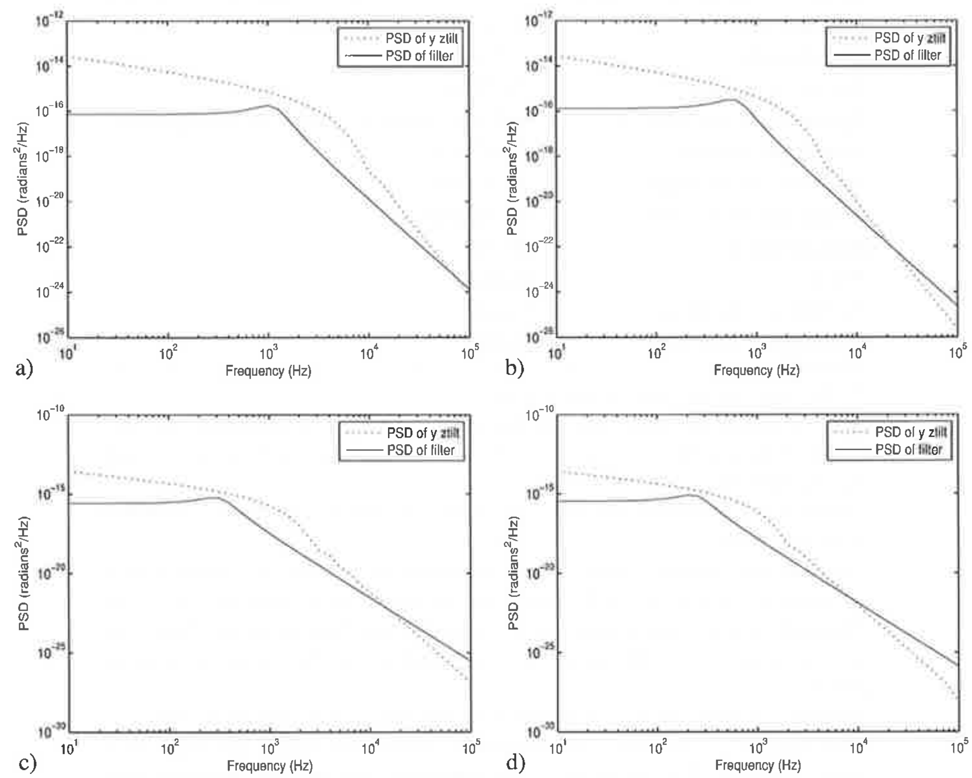

Fig. 1. PSDs of x and y ztilt for apertures of a) 1, b) 2, c) 3.5, and d) 5 m.

A key element in developing a Kalman filter is estimating the power spectral density (PSD) of the turbulence being modeled to shape the stochastic noise used in the filter.[5] A MATLAB® toolbox, ATMtools,[6] is able to give us an approximation for the ztilt's temporal PSD using the function ZTiltPSD(). This function takes inputs of the atmosphere and engagement profile (where everything is and what speed it is going) and returns the ztilt PSD. The PSDs are shown in Fig. 1 for the apertures and conditions applicable to these conditions. As expected, the bandwidths of the PSDs are inversely proportional to the system's aperture diameter.

2.5. System description

The system is modeling communication with an LEO satellite. The relevant parameters are shown in Table 1.

Table 1. Relevant parameters

Parameter	Description
Height of orbit	500 km (circular)
Number of screens	10
Height of turbulence screens	0, 0.4, 1.1, 2.5, 4.5, 7, 10, 12, 15, 20 km
C^2 values	Based on HV5/7 model[a]
Wavelength	1.0 μm
Speed of satellite	7,619 m/s[b]
Speed of ground station	464 m/s (ground station at the equator)
Speed differential	7,155 m/s[c]
Satellite zenith angle	At zenith
Frame rate of system	10^4 frames/s
Run duration	50 ms
Wind	None[a]
Turbulence pixel size	1 cm[d]
Turbulence grid size	2,048[e]

[a]Effective r_0 for the system was 5.4 cm.
[b]For simulation purposes, the satellite is considered to be stationary still and pseudo wind is used to compensate for how the transmission path moves through the atmosphere.
[c]Satellite is in an equatorial orbit, so the speed differential is simply the difference of the two speeds.
[d]Nominal calculations of required pixel sizes indicated that pixels less than 4.8 cm are adequate. In practice, both 4 and 2 cm yielded result deemed too "noisy" to effectively implement the ideal linear estimation that Fried proposed. Pixel size of 1 cm yielded essentially noise-free results conducive to the required estimation process.
[e]Turbulence grid size was determined so that the fastest moving screen was large enough to cover the entire simulation period. The fastest moving screen moves just over 14 m during the 0.05-s simulation, and specified pixel and grid parameters make the screens more than 20 m wide.

3. Simulation Results

The simulation parameters were input to the WaveTrain® simulation program. The program yielded 2,048 $\times$ 2,048 complex arrays of the field at the aperture of the ground station. Of the data, only the central 504 $\times$ 504 pixels were stored because the largest aperture being examined was 5 m in diameter. Even with reduced pixel storage, each frame required 2 MB of storage, so the simulation was divided into 10 subsimulations of 50 frames each. Each subsimulation required approximately 1 h of computation time. Rather than generating a suitable number of runs to perform Monte Carlo analysis, a single set of data was generated. The objective of this effort was to demonstrate the potential effectiveness of a Kalman filter, and this can be shown with a single realization. Monte Carlo analysis is left for future work.

The ztilt of each frame was computed using MATLAB®. The first step was to create an array of angles from the field. The next step was to unwrap the angles to eliminate 2π

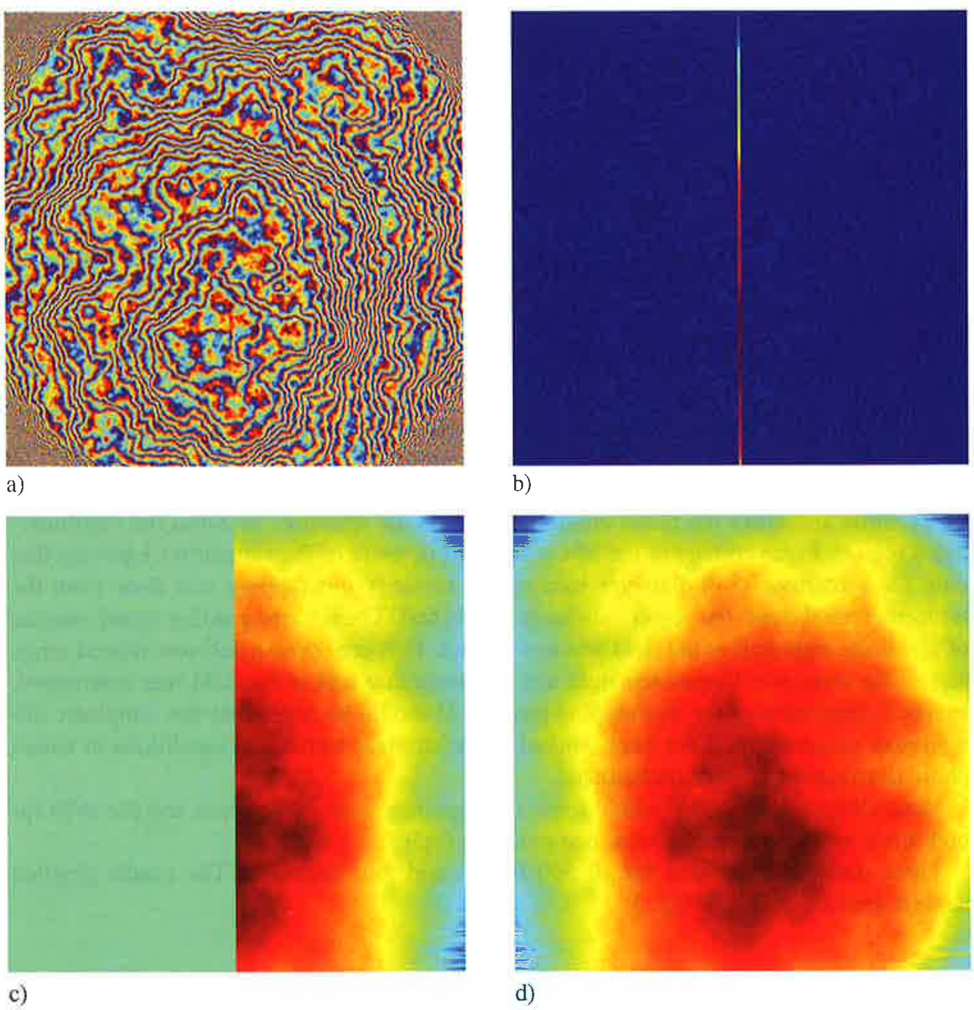

Fig. 2. Unwrapping process: a) wrapped, b) center column unwrapped, c) half unwrapped, and d) fully unwrapped.

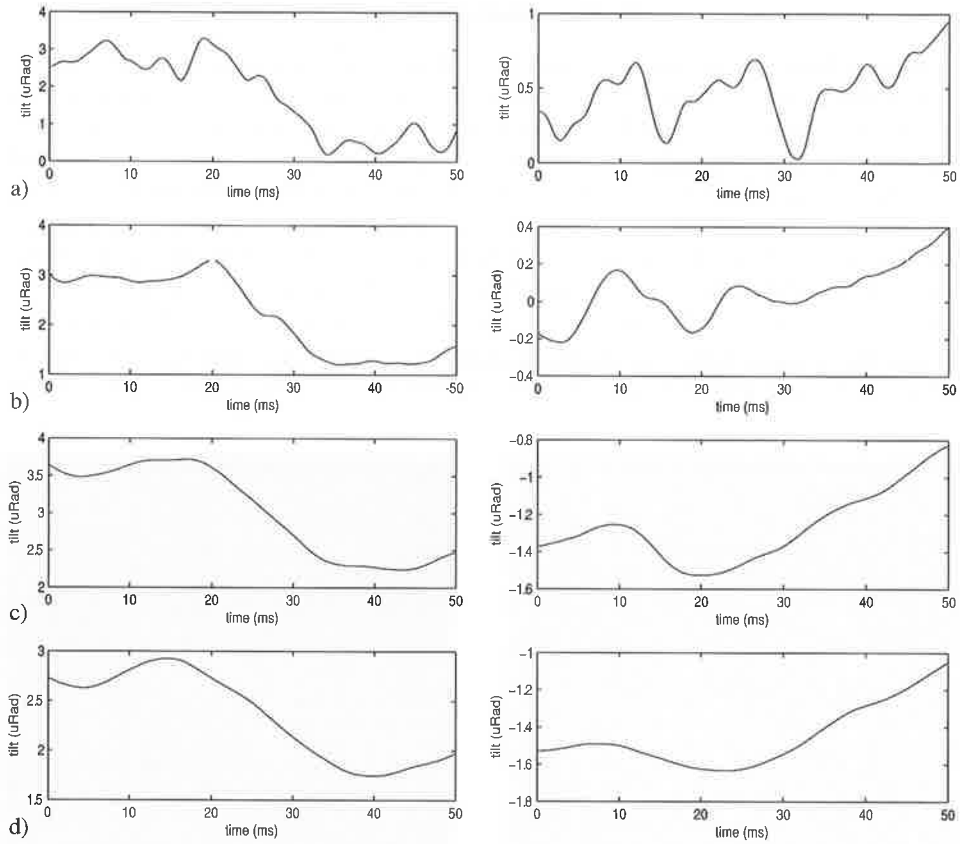

Fig. 3. x (left) and y (right) ztilts for apertures of a) 1, b) 2, c) 3.5, and d) 5 m.

phase shifts and make the phase continuous across the aperture. To avoid the nonilluminated regions in the corners of the 504 × 504 grid (a result of the simulation, knowing that only the centermost 5-m-diameter section was needed), unwrapping was done from the top/center pixel down the center column of pixels first. Then, starting at that center column of pixels the right half of the field was unwrapped. This generated a half-unwrapped array, and so the array was flipped left/right and the remaining half of the field was unwrapped. Figure 2 demonstrates the unwrapping process. It should be noted that this simplistic unwrapping process would not have worked under stronger turbulence conditions in which phase unwrapping is more problematic.

Next, all the pixels falling within a specified aperture were determined, and the ztilts for both the x and y orientations were computed by Eq. (3).[2]

These data were repeated for all 500 frames and four apertures. The results graphed against time are shown in Fig. 3.

4. Kalman Filter Implementation

The key to a Kalman filter is building a good model of the system. Ideally this implies matching the PSD of the process and the PSD of the stochastic noise model exactly, but in

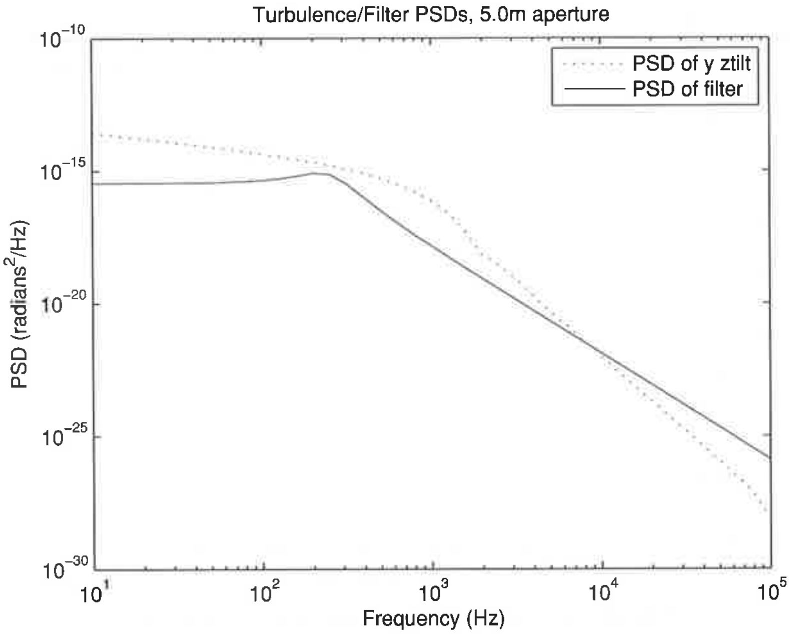

Fig. 4. PSD of filter noise, second-order Gauss–Markov process,[5] with 0.34 damping factor.

practice simpler models are used. For this paper, it is adequate to model the turbulence as a damped second-order Gauss–Markov process. This model uses three states (ztilt, d/dt ztilt, and $\mathrm{d}^2/\mathrm{d}t^2$ ztilt) as shown next:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \text{ztilt} \\ \frac{\mathrm{d}}{\mathrm{d}t}\text{ztilt} \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\text{ztilt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\begin{bmatrix} \text{ztilt} \\ \frac{\mathrm{d}}{\mathrm{d}t}\text{ztilt} \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\text{ztilt} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t),$$

where $w(t)$ is white, Gaussian, zero-mean noise; ω_n sets the bandwidth of the noise; and ζ is the damping factor that controls the shape of the PSD roll-off around the bandwidth frequency.

Having chosen a noise type, the strength $w(t)$, bandwidth ω_n, and damping factor ζ of the noise are varied during the tuning process to maximize performance. Noise driven by a damping factor of 0.34 was optimal for all apertures. The equations are set up to keep the filter energy constant, so that once the noise strength for one aperture is determined, only ω_n is varied to produce a minimum mean-squared error (MSE) (presented momentarily) for the other apertures. The PSDs of the damped second-order Gauss–Markov process and turbulence are shown in Fig. 4.

As expected, the optimum ω_n was approximately inversely proportional to the aperture diameter. Filter bandwidth versus aperture is plotted in Fig. 5.

5. Estimation Results

The estimation performance was computed for the three cases of no estimation (baseline), simple linear estimation, and estimation using a Kalman filter. The plots of ztilts produced

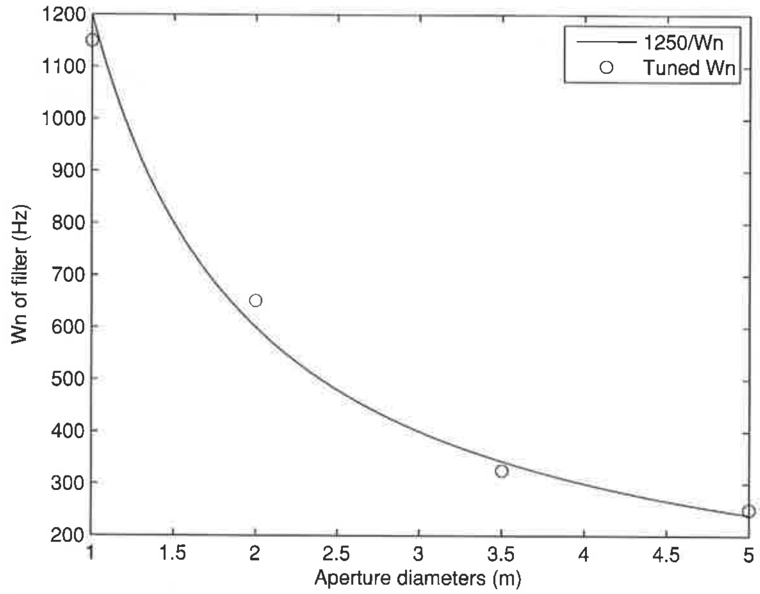

Fig. 5. w_n of tuned filters versus aperture diameter.

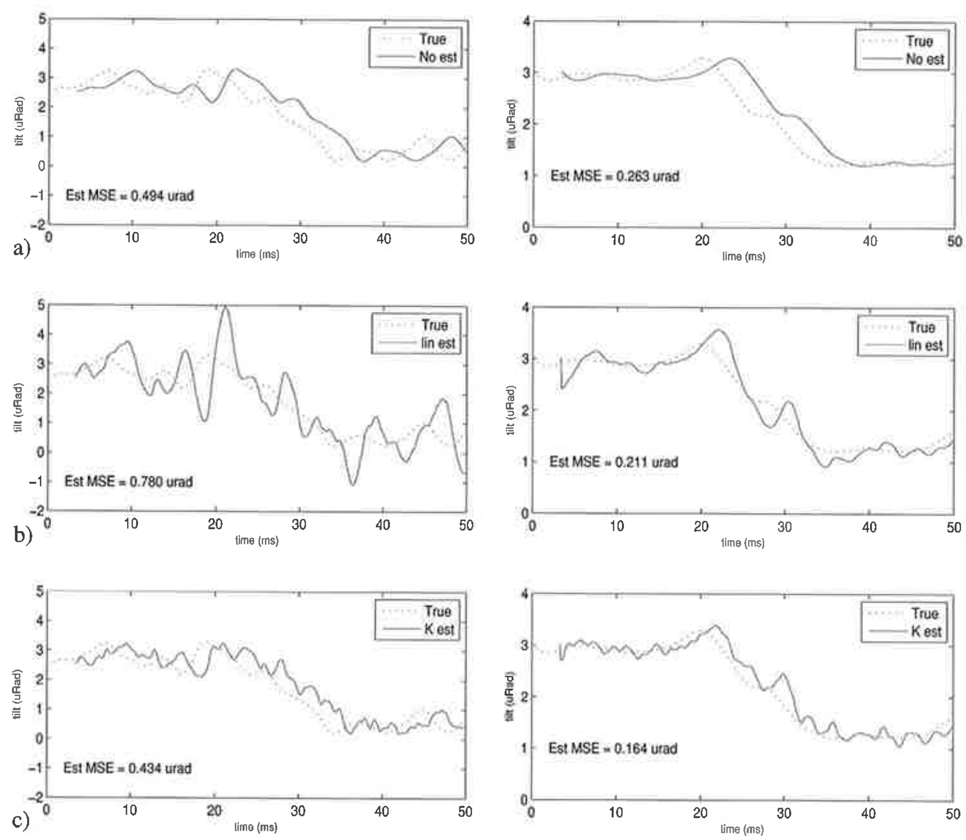

Fig. 6. No estimate (a), linear estimates (b), and Kalman estimates (c) for apertures of 1 (left) and 2 (right) m.

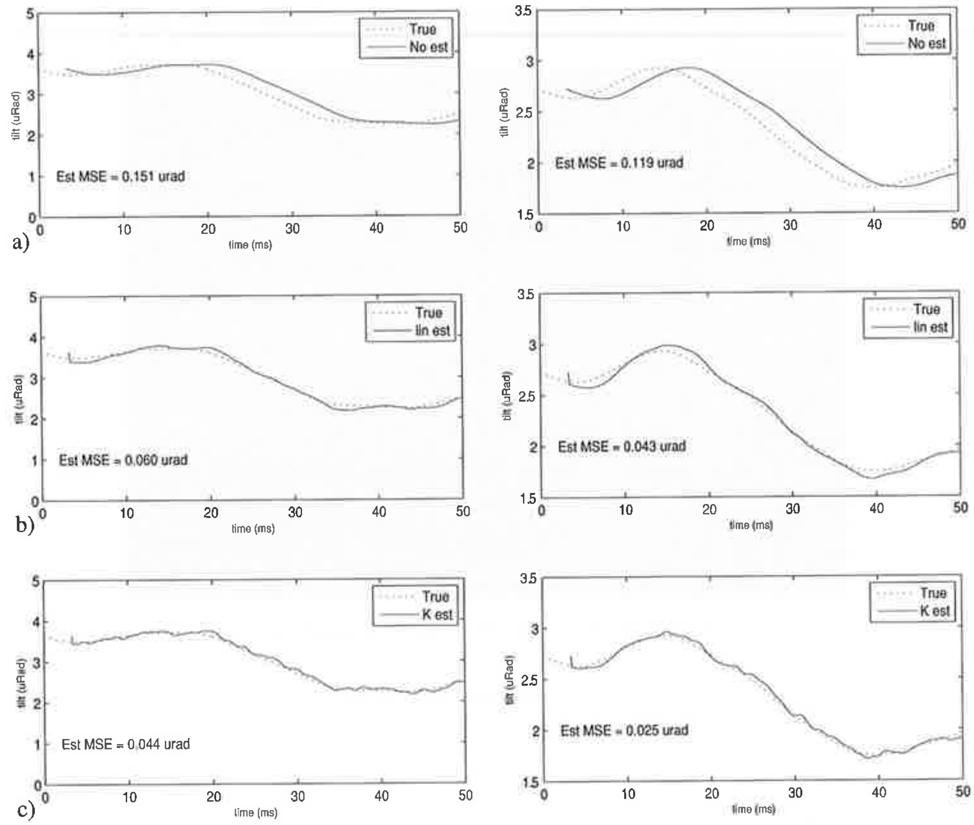

Fig. 7. No estimate (a), linear estimates (b), and Kalman estimates (c) for apertures of 3.5 (left) and 5 (right) m.

in the simulation were treated as truth data, and estimations were produced at each point (time epochs 1–467) for a point 33 time epochs into the future. The baseline case of using no estimation simply shifts the ztilt 3.3 ms into the future. The estimations for the baseline case of no estimation, linear estimation, and Kalman estimation are shown for 1.0-, 2.0-, 3.5-, and 5.0-m apertures in Figs. 6 and 7.

The error is computed for each estimation method as the MSE between the truth data and each method's estimate of the ztilt for time epochs (frames) 34–500. Frame 34 was the first frame with an estimate, and frame 500 was the last frame with truth data. Estimators with lower MSEs were deemed more effective. The results are shown in Table 2.

Table 2. MSEs of each estimation method versus aperture diameter

Estimation	1-m aperture	2-m aperture	3.5-m aperture	5-m aperture
None	0.494 μrad	0.263 μrad	0.151 μrad	0.119 μrad
Linear	0.780 μrad	0.211 μrad	0.060 μrad	0.043 μrad
Kalman	0.434 μrad	0.164 μrad	0.044 μrad	0.025 μrad

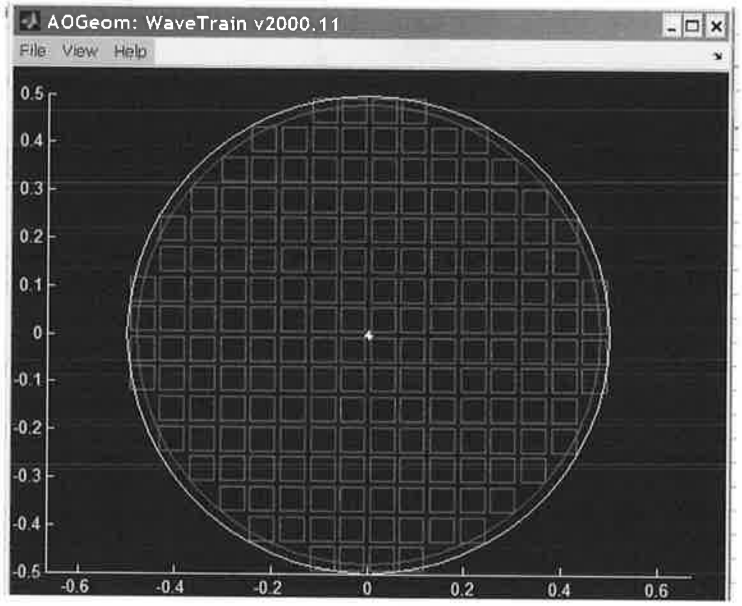

Fig. 8. 6-cm subapertures in 1-m aperture.

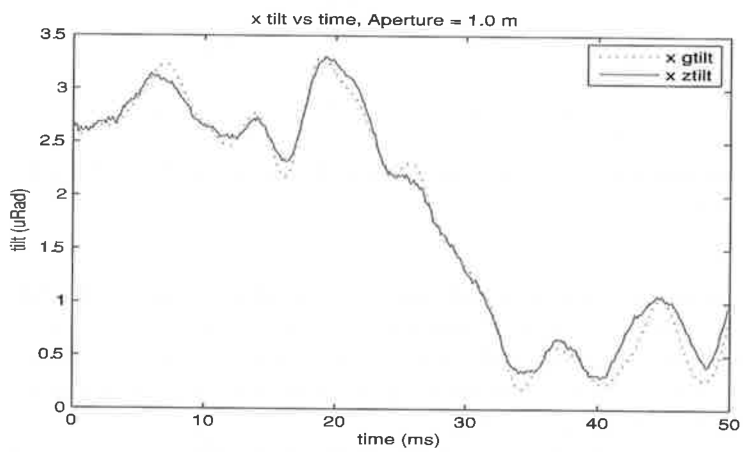

Fig. 9. Estimated ztilt from gtilt data.

Both estimation methods show dramatic improvement over the case of using no estimation at the larger diameters, and the Kalman filter further reduces errors significantly (>50% in the 5-m aperture case). At the smaller diameters, both estimation methods become less effective because the higher frequency content (Fig. 1) of the ztilt reduces the capability of any estimator to perform estimation for a fixed look-ahead.

In effect, as the PSD of a signal gets wider, the signal's autocorrelation gets narrower,[5] lessening look-ahead capability. A Kalman filter will still be optimum in that it should never be worse than not taking an estimate at all. This is shown in that for a 1-m aperture, linear estimation is actually detrimental to the system while the Kalman filter still shows 12% improvement over the baseline case of no estimation.

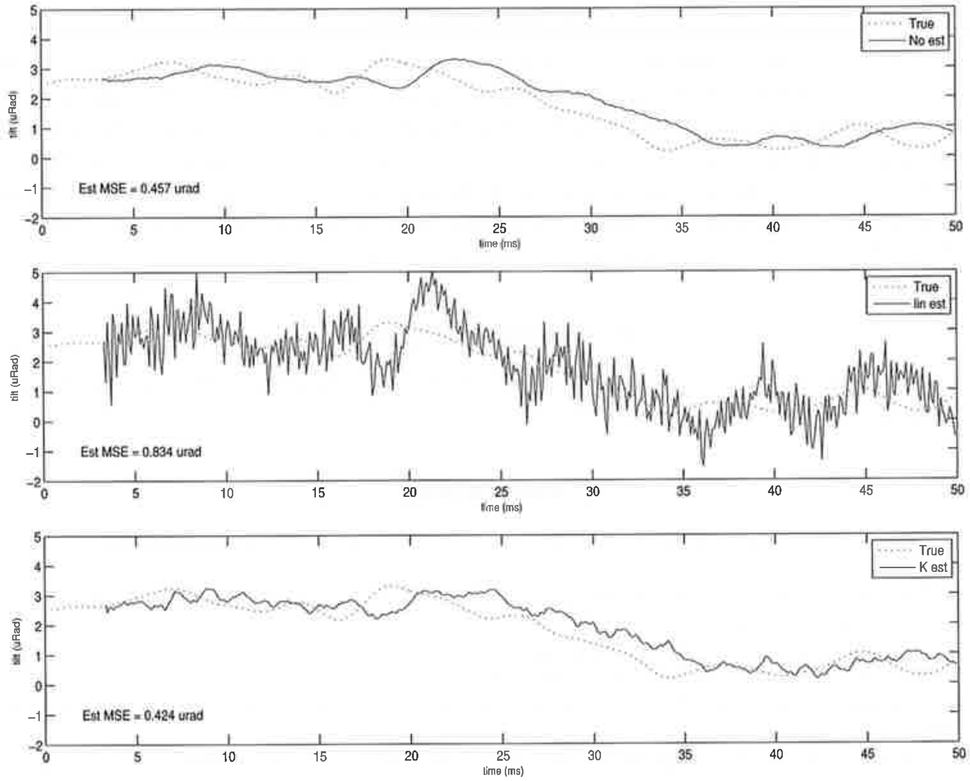

Fig. 10. Estimating ztilt from gtilt data; aperture = 1 m.

6. Extrapolation to gtilt

To assess the potential of ztilt estimation using a Kalman filter under more realistic conditions, a WFS was simulated instead of the previously simulated field sensor for the 1-m-aperture case. The 1-m aperture was divided into 188 subapertures (6×6 cm each) as depicted in Fig. 8.

The resulting gtilt is plotted against the idealized ztilt generated from the field sensor in Fig. 9.

The resulting plot of gtilt is essentially a noisy version of ztilt. The noise comes from several limitations of S–H sensors but is dominated by the D_{SA}/r_0 ratio for the subaperture, which is still big enough that the ctilt for each specific subaperture is not equivalent to the ztilt for that subaperture.

As before, ztilt was estimated 3.3 ms ahead using the output of the S–H WFS. The results are shown in Fig. 10.

In this case, the Kalman filter modeled the measurement as noisy to account for the difference between the measured gtilt and the "truth" ztilt. The results of no estimation, linear estimation, and Kalman estimation are depicted in Fig. 10. The Kalman filter still outperforms the no-estimate case. As in the idealized case where ztilt was available to the estimators, the improvement is minimal as the 1-m aperture causes the temporal frequency content of turbulence to be too high.

7. Conclusions

A Kalman filter is shown as an effective estimator of anisoplanatic ztilt under both idealized conditions, in which ztilt is available to the estimator, and the more realistic case, in which only gtilt is available. Future work should include noise sources in the sensors, varying subaperture sizes, varying aperture sizes (in the gtilt case), and performing Monte Carlo analysis on the system to tune the Kalman filter. In addition, future work should quantify the performance enhancements of improved anisoplanatic ztilt estimation in terms of Strehl ratios and signal strength for a nominal receiver.

8. Acknowledgment

The opinions and views expressed by the authors are not necessarily those of the Department of Defense or the U.S. Air Force.

References

[1]Andrews, L.C., and R.C. Philips, *Laser Beam Propagation through Random Media*, SPIE Optical Engineering Press (1998).

[2]Fried, D.L., "Report no. tn-194, Tilt-Anisoplanatism Amelioration Algorithm: Evaluation Using a Propagation Simulation Data Set," May 2005.

[3]Fried, D., Personal Communication, September 2006.

[4]Goodman, J.W., *Statistical Optics*, Wiley-Interscience (1985).

[5]Maybeck, P.S., *Stochastic Modelling, Estimation and Control*, Volume 1, Navtech Book & Software Store (1994).

[6]MZA, "A Toolbox for Atmospheric Propagation Modeling," Software Version 2000.11.

The Authors

Dr. Juan R. Vasquez received the B.S. degree in electrical engineering from Oklahoma State University in 1987. He received the M.S. and Ph.D. degrees in electrical engineering from the Air Force Institute of Technology (AFIT) in 1992 and 1998. He is a Lieutenant Colonel in the U.S. Air Force serving on the faculty at AFIT. His military service includes assignments as an aircraft maintenance officer, program manager for weather satellite systems, researcher in global positioning system antijam techniques, and program manager for a multitarget, multisensor fusion program. Dr. Vasquez's research interests include target tracking, navigation, and multiple-model estimation and control.

Mr. Todd M. Venema received the B.S. degree in engineering from Calvin College in 1988. He received the M.S. in electrical engineering from North Carolina State University in 1990. He is a Lieutenant Colonel in the U.S. Air Force currently pursuing his Ph.D. degree at the Air Force Institute of Technology. His military service includes duty as global positioning system satellite engineer, KC-135R pilot, and B-52H test pilot. Lt. Col. Venema's doctoral research involves studying closed-loop control for an adaptive optics system utilizing a self-referencing interferometer and continuous facesheet deformable mirror.